

# Local existence and uniqueness of solutions for non stationary compressible viscoelastic fluid of Oldroyd type

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## Abstract

This work is devoted to the study of a compressible viscoelastic fluids satisfying the Oldroyd-B model in a regular bounded domain. We prove the local existence of solutions and uniqueness of flows by a classical fixed point argument.

## 1 Introduction

In this paper, we study the local existence of solutions for compressible viscoelastic fluid flows in the case of the Oldroyd-B model in a regular bounded domain in  $\mathbb{R}^3$ . We also show the uniqueness of solutions. We prove l'existence by using the classical method based on the Schauder fixed-point theorem. Valli, in [8], show the local existence in the case of the Navier-Stokes equations. The case of the Oldroyd model for incompressible fluid is studied by Guillopé and Saut in [3]. Talhouk shows the existence and the uniqueness for Jeffreys model's in [6].

This paper is organized as follows. Section 2 is devoted to the modeling of the problem and to the definition of well-prepared initial conditions. The principal notation and results are detailed in Section 3. The local existence of regular solutions is given in Section 4.

## 2 The Modeling

### 2.1 Unsteady Flows of Compressible Viscoelastic Fluids

Consider unsteady flows of viscoelastic fluids in a bounded domain  $\Omega^*$  of  $\mathbb{R}^3$  with a regular boundary  $\Gamma^*$ . The system, obtained from the laws of conservation of momentum, and of mass, and from the constitutive equation of the fluid, reads as follows [4]: in  $Q_{T^*}^* = (0, T^*) \times \Omega^*$ ,

$$\left\{ \begin{array}{l} \rho^* \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) = \rho^* \mathbf{f}^* + \operatorname{div}^* (\tau^* - p^* \mathbf{I}), \\ \frac{\partial \rho^*}{\partial t^*} + \operatorname{div}^* (\rho^* \mathbf{u}^*) = 0, \\ \tau^* + \lambda \frac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} = 2\eta \left( \mathbf{D}^* + \mu \frac{\mathcal{D}_a \mathbf{D}^*}{\mathcal{D} t^*} \right). \end{array} \right. \quad (2.1)$$

The  $*$ -variables are the dimensional ones in the domain of the flow  $\Omega^*$ , and  $T^* > 0$  is a dimensional time. The unknowns are the velocities  $\mathbf{u}^*$ , the density  $\rho^*$ , and the symmetric tensor of constraints  $\tau^*$ .  $\eta$  is the total viscosity of the fluid,  $\lambda > 0$  is the relaxation time, and  $\mu$  is the retardation time ( $0 < \mu < \lambda$ ).

$\frac{\mathcal{D}_a \tau^*}{\mathcal{D}t^*}$  is an objective derivative of the tensor  $\tau^*$ , given by

$$\frac{\mathcal{D}_a \tau^*}{\mathcal{D}t^*} = \left( \frac{\partial}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \right) \tau^* + \tau^* \mathbf{W}^* - \mathbf{W}^* \tau^* - a(\mathbf{D}^* \tau^* + \tau^* \mathbf{D}^*),$$

where  $\mathbf{W}^* = \mathbf{W}^*[\mathbf{u}^*] = \frac{1}{2}(\nabla^* \mathbf{u}^* - \nabla^{*\top} \mathbf{u}^*)$  and  $\mathbf{D}^* = \mathbf{D}^*[\mathbf{u}^*] = \frac{1}{2}(\nabla^* \mathbf{u}^* + \nabla^{*\top} \mathbf{u}^*)$  are, respectively, the rate of rotation and the rate of deformation tensors.  $a$  is a real parameter in  $[-1, 1]$ .

System (2.1) is completed by a condition on the boundary,

$$\mathbf{u}^* = 0 \text{ on } \Sigma_{T^*}^* = (0, T^*) \times \Gamma^*,$$

and by the initial data

$$\mathbf{u}^*(0, \cdot) = \mathbf{u}_0^*, \quad \rho^*(0, \cdot) = \rho_0^*, \quad \tau^*(0, \cdot) = \tau_0^*, \quad \text{in } \Omega^*.$$

We split  $\tau^*$  into two parts: the Newtonian one  $\tau_s^*$  related to the solvent, and the polymeric one  $\tau_p^*$ . We may write

$$\tau^* = \tau_s^* + \tau_p^* = 2\eta_s \mathbf{D}^* + \tau_e^*,$$

where  $\tau_e^* = \tau_p^* - \left( \frac{2\xi_s}{3} \text{div}^* \mathbf{u}^* \right) \mathbf{I}$ , and  $\mathbf{I}$  is the identity tensor.  $\eta_s = \eta\mu/\lambda$  and  $\xi_s$  are the solvent viscosity and the group viscosity, respectively. Since we are interested in a model for weakly compressible fluids, we suppose that  $\xi_s = 0$ . From the third equation in (2.1), we can deduce that  $\tau_e^*$  satisfies the equation

$$\tau_e^* + \lambda \frac{\mathcal{D}_a \tau_e^*}{\mathcal{D}t^*} = 2\eta_e \mathbf{D}^*,$$

where  $\eta_e = \eta - \eta_s$  is called the polymer viscosity.  $\eta_s$  and  $\eta_e$  are two non-negative numbers.

Therefore, under the assumption  $\xi_s = 0$ , System (2.1) is equivalent to the system in  $\mathbf{Q}_{T^*}^*$ ,

$$\left\{ \begin{array}{l} \rho^* \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) = \rho^* \mathbf{f}^* + \eta_s (\Delta^* \mathbf{u}^* + \nabla^* \text{div}^* \mathbf{u}^*) - \nabla^* p^* + \text{div}^* \tau^*, \\ \frac{\partial \rho^*}{\partial t^*} + \text{div}^* (\rho^* \mathbf{u}^*) = 0, \\ \tau^* + \lambda \frac{\mathcal{D}_a \tau^*}{\mathcal{D}t^*} = 2\eta_e \mathbf{D}^*[\mathbf{u}^*], \end{array} \right. \quad (2.2)$$

where we have denoted  $\tau_e^*$  by  $\tau^*$  to simplify the notation.

## 2.2 Well-Prepared Initial Conditions

We first define the Mach number  $\varepsilon$  as being the ratio of the typical velocity of the fluid  $U_0$  to the speed of sound  $\left( \frac{dp^*}{d\rho^*}(\bar{\rho}_0^*) \right)^{1/2}$  in the same fluid at the same state. We divide the density

$\rho^* = \rho^{*\varepsilon}$  into two parts: a *constant* one  $\bar{\rho}_0^*$ , independent of  $\varepsilon$ , and a remainder, which is small for small  $\varepsilon's$ , say

$$\rho^{*\varepsilon} = \bar{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \bar{\rho}_0^* + \varepsilon^2 \sigma^{*\varepsilon}.$$

We also suppose that the initial conditions  $\rho_0^{*\varepsilon}$ ,  $\mathbf{u}_0^{*\varepsilon}$  and  $\tau_0^{*\varepsilon}$  are *well-prepared*, which means that they take a similar form, say

$$\begin{aligned} \rho_0^{*\varepsilon} &= \bar{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \bar{\rho}_0^* + \varepsilon^2 \sigma_0^{*\varepsilon}, \\ \mathbf{u}_0^{*\varepsilon} &= \mathbf{v}_0^* + \mathbf{v}_0^{*\varepsilon}, \text{ with } \operatorname{div} \mathbf{v}_0^* = 0, \\ \tau_0^{*\varepsilon} &= \mathbf{S}_0^* + \mathbf{S}_0^{*\varepsilon}, \end{aligned}$$

where  $\mathbf{v}_0^*$  and  $\mathbf{S}_0^*$  are, respectively, a vector and a symmetric tensor, both independent of  $\varepsilon$ .

We assume

$$\mathfrak{m}^* = \min_{\Omega^*} \rho_0^* > 0 \quad \text{and} \quad \mathfrak{M}^* = \max_{\Omega^*} \rho_0^*.$$

Assuming that  $p^* = p^*(\rho^*)$  is regular, say class  $C^3$  at least, we remark

$$\frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + \varepsilon^2 \sigma^*) - \frac{dp^*}{d\rho^*}(\bar{\rho}_0^*) = \varepsilon^2 \int_0^1 \frac{d^2 p^*}{d\rho^{*2}}(\bar{\rho}_0^* + s\varepsilon^2 \sigma^*) ds.$$

We introduce the function  $w^*$ , defined by  $w^*(\sigma^*) = \frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + \varepsilon^2 \sigma^*) - \frac{dp^*}{d\rho^*}(\bar{\rho}_0^*)$ , and remark that  $w^*$  depends on  $\varepsilon$ , satisfies  $w^*(0) = 0$ , and is of class  $C^2$  at least.

Replacing  $\rho^*$  by its value in the first equation (2.2), one infers

$$\begin{aligned} (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) + \varepsilon^2 \frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \nabla^* \sigma^* \\ = (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \mathbf{f}^* + \eta_s (\Delta^* \mathbf{u}^* + \nabla^* \operatorname{div}^* \mathbf{u}^*) + \operatorname{div}^* \tau^*. \end{aligned}$$

We can also rewrite this equality, by taking into account the definitions of  $w^*(\sigma^*)$  and of the Mach number  $\varepsilon$ , in the form

$$\begin{aligned} (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) + (\mathbf{U}_0)^2 \nabla^* \sigma^* \\ = (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \mathbf{f}^* + \eta_s (\Delta^* \mathbf{u}^* + \nabla^* \operatorname{div}^* \mathbf{u}^*) + \operatorname{div}^* \tau^* - \varepsilon^2 w^*(\sigma^*) \nabla^* \sigma^*. \end{aligned}$$

From the second equation in (2.2) we easily deduce

$$\varepsilon^2 \frac{\partial \sigma^*}{\partial t^*} + \bar{\rho}_0^* \operatorname{div}^* \mathbf{u}^* + \varepsilon^2 \operatorname{div}^* (\sigma^* \mathbf{u}^*) = 0.$$

Finally, System (2.2) can be written as follows, in  $\mathbf{Q}_{T^*}^*$ ,

$$\left\{ \begin{aligned} (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) + (\mathbf{U}_0)^2 \nabla^* \sigma^* &= (\bar{\rho}_0^* + \varepsilon^2 \sigma^*) \mathbf{f}^* + \operatorname{div}^* \tau^* \\ &\quad + \eta_s (\Delta^* \mathbf{u}^* + \nabla^* \operatorname{div}^* \mathbf{u}^*) - \varepsilon^2 w^*(\sigma^*) \nabla^* \sigma^*, \\ \varepsilon^2 \frac{\partial \sigma^*}{\partial t^*} + \bar{\rho}_0^* \operatorname{div}^* \mathbf{u}^* + \varepsilon^2 \operatorname{div}^* (\sigma^* \mathbf{u}^*) &= 0, \\ \tau^* + \lambda \frac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} &= 2\eta_e \mathbf{D}^*[\mathbf{u}^*]. \end{aligned} \right. \quad (2.3)$$

### 2.3 Dimensionless Variables

We introduce the dimensionless variables,

$$\mathbf{x}^* = L_0 \mathbf{x}, \quad t^* = \frac{L_0}{U_0} t, \quad \rho^* = a_0 \rho, \quad w^*(\sigma^*) = (U_0)^2 w(\sigma),$$

$$\mathbf{u}^* = U_0 \mathbf{u}, \quad \sigma^* = a_0 \sigma, \quad \tau^* = T_0 \tau, \quad p^*(\rho^*) = T_0 p(\rho), \quad \mathbf{f}^* = \frac{(U_0)^2}{L_0} \mathbf{f},$$

where  $L_0$  represents a typical length of the flow. The real numbers  $a_0 = \frac{\eta}{U_0 L_0}$  and  $T_0 = \frac{\eta U_0}{L_0}$  characterize the density and the stress tensor of the fluid.  $\Omega$  denotes the non-dimensional domain of the flow, with boundary  $\Gamma$ , and  $T > 0$  a non-dimensional time.

We introduce three non-dimensional numbers: a number  $\alpha$  similar to the Reynolds number for incompressible flows, the Weissenberg number  $We$ , and a number  $\omega$  relative to the viscosities of the fluid,

$$\alpha = \frac{\bar{\rho}_0^*}{a_0} = \frac{\bar{\rho}_0^* U_0 L_0}{\eta}, \quad We = \frac{\lambda U_0}{L_0}, \quad \omega = 1 - \frac{\eta_s}{\eta}.$$

We also define

$$w(\sigma) = \alpha \left\{ \frac{dp}{d\rho}(\alpha + \varepsilon^2 \sigma) - \frac{dp}{d\rho}(\alpha) \right\}.$$

In dimensionless variables, System (2.3) takes the form, in  $Q_T = (0, T) \times \Omega$ ,

$$\left\{ \begin{array}{l} \mathbf{u}' + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\alpha + \varepsilon^2 \sigma} \nabla \sigma = \mathbf{f} + \frac{1 - \omega}{\alpha + \varepsilon^2 \sigma} (\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u}) + \frac{\operatorname{div} \tau}{\alpha + \varepsilon^2 \sigma} \\ \quad - \frac{\varepsilon^2 w(\sigma) \nabla \sigma}{\alpha + \varepsilon^2 \sigma}, \\ \sigma' + \varepsilon^{-2} \alpha \operatorname{div} \mathbf{u} + \operatorname{div}(\sigma \mathbf{u}) = 0, \\ \tau + We \{ \tau' + (\mathbf{u} \cdot \nabla) \tau + \mathbf{g}(\nabla \mathbf{u}, \tau) \} = 2\omega \mathbf{D}[\mathbf{u}], \end{array} \right. \quad (2.4)$$

with the notation  $\mathbf{u}' = \frac{\partial \mathbf{u}}{\partial t}$ ,  $\sigma' = \frac{\partial \sigma}{\partial t}$  and  $\tau' = \frac{\partial \tau}{\partial t}$ , and

$$\mathbf{g}(\nabla \mathbf{u}, \tau) = \tau \mathbf{W}[\mathbf{u}] - \mathbf{W}[\mathbf{u}] \tau - a(\mathbf{D}[\mathbf{u}] \tau + \tau \mathbf{D}[\mathbf{u}]).$$

Introducing the differential operator  $A = -(\Delta + \nabla \operatorname{div})$  we may rewrite System (2.4) as follows, in  $Q_T$ ,

$$\left\{ \begin{array}{l} \alpha (\mathbf{u}' + (\mathbf{u} \cdot \nabla) \mathbf{u}) + (1 - \omega) A \mathbf{u} + \nabla \sigma = \mathbf{F}(\mathbf{u}, \sigma, \tau) + \operatorname{div} \tau, \\ \sigma' + (\mathbf{u} \cdot \nabla) \sigma + \sigma \operatorname{div} \mathbf{u} = -\varepsilon^2 \alpha \operatorname{div} \mathbf{u}, \\ \tau + We (\tau' + (\mathbf{u} \cdot \nabla) \tau + \mathbf{g}(\nabla \mathbf{u}, \tau)) = 2\omega \mathbf{D}[\mathbf{u}], \end{array} \right. \quad (2.5)$$

with

$$\mathbf{F}(\mathbf{u}, \sigma, \tau) = \alpha \mathbf{f} + \frac{(1 - \omega) \varepsilon^2 \sigma}{\alpha + \varepsilon^2 \sigma} A \mathbf{u} + \frac{\varepsilon^2 (\sigma - w(\sigma))}{\alpha + \varepsilon^2 \sigma} \nabla \sigma - \frac{\varepsilon^2 \sigma}{\alpha + \varepsilon^2 \sigma} \operatorname{div} \tau. \quad (2.6)$$

System (2.5) is completed by an homogeneous condition on the boundary,

$$\mathbf{u} = 0 \text{ on } \Sigma_T = (0, T) \times \Gamma, \quad (2.7)$$

and by three initial conditions,

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \sigma(0, \cdot) = \sigma_0, \quad \tau(0, \cdot) = \tau_0, \quad \text{in } \Omega. \quad (2.8)$$

We also assume the followings,

$$0 < \mathfrak{m}_1 = \frac{\mathfrak{m}^*}{a_0} \leq \alpha + \varepsilon^2 \sigma_0 \leq \mathfrak{M}_1 = \frac{\mathfrak{M}^*}{a_0}, \quad \text{in } \Omega,$$

where  $\mathfrak{m}_1$  and  $\mathfrak{M}_1$  are some given constants.

### 3 The Notation and Main Results

#### 3.1 Notation

$\Omega$  is a bounded domain in  $\mathbb{R}^3$ , with a regular boundary  $\Gamma$ , and  $\mathbf{n}$  denotes the unit outward-pointing normal vector to  $\Gamma$ . For  $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ , we denote by  $|\mathbf{x}|$  its Euclidean norm.

We will use the following spaces: the Lebesgue spaces  $L^p(\Omega)$ ,  $1 \leq p \leq +\infty$ , with norms  $\|\cdot\|_{L^p}$  (except for the  $L^2(\Omega)$ -norm, which is denoted by  $\|\cdot\|$ ); the Sobolev space  $H^k(\Omega)$ ,  $k \in \mathbb{N}^*$ , with norm  $\|\cdot\|_k$  and inner product  $((\cdot, \cdot))_k$ ; the vector spaces  $\mathbf{L}^2(\Omega)$  and  $\mathbf{H}^k(\Omega)$  of vector-valued or tensor-valued functions with components in  $L^2(\Omega)$  and  $H^k(\Omega)$  respectively, their norms being denoted in the same way as above. We will also use the homogeneous Sobolev space  $H_0^1(\Omega)$  and its dual  $H^{-1}(\Omega)$ .

If  $I$  is an interval of  $\mathbb{R}_+$  and  $k \in \mathbb{N}$ ,  $C(\bar{I}; \mathbf{H}^k(\Omega))$  is the space of vector- or tensor-valued functions which are continuous on  $\bar{I}$  with values in  $\mathbf{H}^k(\Omega)$ . The norm, in this space, is denoted by  $\|\cdot\|_{C,k}$ .  $C_b(\bar{I}; \mathbf{H}^k(\Omega))$  is the space of functions of  $C(I; \mathbf{H}^k(\Omega))$  which are bounded on  $\bar{I}$ .

The space  $L^p(I; \mathbf{H}^k(\Omega))$ , for  $1 \leq p \leq +\infty$ , and  $k \in \mathbb{N}$ , consists of  $p$ -integrable functions on  $I$  with values in  $\mathbf{H}^k(\Omega)$ . For  $1 \leq p \leq +\infty$ ,  $k \in \mathbb{N}$  and  $0 < T \leq \infty$ , the norm in  $L^p((0, T), \mathbf{H}^k(\Omega))$  is denoted by  $[\cdot]_{p,k,T}$ .  $L_{\text{loc}}^2(\mathbb{R}_+; \mathbf{H}^k(\Omega))$  is the set of functions which are in  $L^2(I; \mathbf{H}^k(\Omega))$  for all bounded interval  $I$  in  $\mathbb{R}_+$ .

The letters  $C$ ,  $c_i$  or  $c_i^j$ ,  $i, j = 1, 2, \dots$ , will denote constants taking different values, but not depending on  $\varepsilon$ .  $C_\Omega$  will be a constant, taking different values, and depending only on  $\Omega$ .  $(2.1)_n$  denotes the  $n$ -th equation of System (2.1).

#### 3.2 The Main Result

Recall the problem under study:

$$\left\{ \begin{array}{ll} \alpha(\mathbf{u}' + (\mathbf{u} \cdot \nabla)\mathbf{u}) + (1 - \omega)A\mathbf{u} + \nabla\sigma &= \mathbf{F}(\mathbf{u}, \sigma, \tau) + \text{div } \tau, \\ \sigma' + (\mathbf{u} \cdot \nabla)\sigma + \sigma \text{div } \mathbf{u} &= -\varepsilon^2 \alpha \text{div } \mathbf{u}, \\ \tau + \text{We}(\tau' + (\mathbf{u} \cdot \nabla)\tau + \mathbf{g}(\nabla\mathbf{u}, \tau)) &= 2\omega\mathbf{D}[\mathbf{u}], \quad \text{in } Q_T, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0, \quad \text{in } \Omega, \\ \sigma(0, \cdot) &= \sigma_0, \quad \text{in } \Omega, \\ \tau(0, \cdot) &= \tau_0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \Sigma_T, \end{array} \right. \quad (3.1)$$

where  $\mathbf{F}$  is defined by (2.6).

**Theorem 3.1.** (Existence of a local solution) *Assume  $\Omega \subset \mathbb{R}^3$  is a domain of class  $C^3$ . Let  $\mathfrak{m}_1$  and  $\mathfrak{M}_1$  be two real constants such that  $0 < \mathfrak{m}_1 \leq \mathfrak{M}_1$ . Assume*

$$\mathbf{f} \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^1(\Omega)), \text{ with } \mathbf{f}' \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbf{H}^{-1}(\Omega)), \mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \tau_0 \in \mathbf{H}^2(\Omega),$$

$$\sigma_0 \in H^2(\Omega), \text{ with } \int_{\Omega} \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \text{ and } 0 < \mathfrak{m}_1 \leq \alpha + \varepsilon^2 \sigma_0 \leq \mathfrak{M}_1, \text{ in } \Omega.$$

*Then there exists a time  $T_1 > 0$  and a solution  $(\mathbf{u}, \sigma, \tau)$  of Problem (2.5)-(2.8) in  $Q_{T_1} = (0, T_1) \times \Omega$ , satisfying*

$$\begin{aligned} \mathbf{u} &\in L^2(0, T_1; \mathbf{H}^3(\Omega)) \cap C([0, T_1]; \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)), \\ \mathbf{u}' &\in L^2(0, T_1; \mathbf{H}_0^1(\Omega)) \cap C([0, T_1]; \mathbf{L}^2(\Omega)), \\ (\tau, \sigma) &\in C([0, T_1]; \mathbf{H}^2(\Omega) \times H^2(\Omega)), \quad (\tau', \sigma') \in C([0, T_1]; \mathbf{H}^1(\Omega) \times H^1(\Omega)), \end{aligned}$$

with

$$\int_{\Omega} \sigma(\cdot, \mathbf{x}) d\mathbf{x} = 0, \quad \text{in } [0, T_1], \text{ and } \frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2 \sigma \leq 2\mathfrak{M}_1, \quad \text{in } \overline{Q}_{T_1}.$$

**Theorem 3.2.** (Uniqueness of a local solution) *There exist a unique solution of Problem (2.5)-(2.8), given in Theorem 3.1.*

To show that the local solution found in Theorem 3.1 exists for all times under certain regularity and smallness conditions on the data, we also assume that the function  $w \in C^2(\mathbb{R})$  has the following properties: for all  $h \in L^2(0, T; H^2(\Omega))$ ,

$$\begin{aligned} \|(w(h))'\| &\leq C \|h'\|, \quad \|w(h)\| \leq C \|h\|, \\ \|w(h)\|_k &\leq C \|h\|_k, \quad k = 1, 2, \end{aligned} \tag{3.2}$$

for some constant  $C$  depending on  $\Omega$  and  $w$ .

**Remark 3.3.** *There are several examples of functions  $p = p(\rho)$ , for which  $w$  satisfies the conditions above. Let us quote the case where the pressure is given by the linear state law  $p(\rho) = \frac{1}{\varepsilon^2}(\rho - \alpha)$ , as well as the case of isothermal compressible perfect fluids, where  $p(\rho) = (C_s)^2 \rho$ , and  $C_s$  is the velocity of sound in the fluid.*

## 4 Existence and Uniqueness of Local Solutions

We prove Theorem 3.1 by using the classical method based on the Schauder fixed-point theorem. To do that in our case, we study three linear problems: the first one has the velocity  $\mathbf{u}$  as unknown, and the next ones are two transport equations for the density  $\sigma$  and for the stress tensor  $\tau$  respectively. The parameter  $\varepsilon$  is fixed in the interval  $(0, 1]$ .

Let  $\mathbf{w}$ ,  $\pi$  and  $\psi$  a given vector, function and the symmetric tensor of constraints respectively. Let  $T$  a positive real number,  $Q_T = \Omega \times ]0, T[$  and  $\Sigma_T = \partial\Omega \times ]0, T[$ . Consider the linear problem,

$$\left\{ \begin{array}{rcl} \alpha \mathbf{u}' + (1 - \omega) A \mathbf{u} & = & \mathfrak{F}, \\ \sigma' + (\mathbf{w} \cdot \nabla) \sigma + \sigma \operatorname{div} \mathbf{w} & = & \mathcal{G}, \\ \tau + \operatorname{We} \{ \tau' + (\mathbf{w} \cdot \nabla) \tau + \mathbf{g}(\nabla \mathbf{w}, \tau) \} & = & 2\omega \mathbf{D}[\mathbf{w}], \quad \text{in } Q_T, \\ \mathbf{u}(0, x) & = & \mathbf{u}_0(x), \\ \sigma(0, x) & = & \sigma_0(x), \\ \tau(0, x) & = & \tau_0(x), \quad \text{in } \Omega, \\ \mathbf{u} & = & 0, \quad \text{on } \Sigma_T, \end{array} \right. \quad (4.1)$$

with

$$\mathfrak{F} = \mathbf{F}(\mathbf{w}, \pi, \psi) - \alpha(\mathbf{w} \cdot \nabla) \mathbf{w} - \nabla \pi + \mathbf{div} \psi, \quad (4.2)$$

$$\mathcal{G} = -\varepsilon^{-2} \operatorname{div} \mathbf{w}. \quad (4.3)$$

and

$$\frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2 \pi \leq 2\mathfrak{M}_1, \quad \text{in } \overline{Q}_T. \quad (4.4)$$

#### 4.1 Linear problem concerning the velocity $\mathbf{u}$

Consider the linear problem concerning the velocity  $\mathbf{u}$ ,

$$\left\{ \begin{array}{rcl} \alpha \mathbf{u}' + (1 - \omega) A \mathbf{u} & = & \mathfrak{F}, \quad \text{in } Q_T, \\ \mathbf{u}(0, x) & = & \mathbf{u}_0(x), \quad \text{in } \Omega, \\ \mathbf{u} & = & 0, \quad \text{on } \Sigma_T, \end{array} \right. \quad (4.5)$$

where  $A \mathbf{u} = -(\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u})$ ,  $\mathfrak{F}$  and  $\mathbf{u}_0$  are given and  $0 < T \leq +\infty$ .

The first Lemma concerns the existence of a unique solution of (4.5). By classical result of Agmon-Douglis-Nirenberg [1],  $A = -\Delta - \nabla \operatorname{div}$  is a strongly elliptic operator, and generates an analytic semigroup in  $\mathbf{L}^2(\Omega)$  with domain  $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$  (we can see for instance [8]).

**Lemma 4.1.** *Let  $\Omega \subset \mathbb{R}^3$  of class  $C^2$ ,  $\mathfrak{F} \in L^2(0, T; \mathbf{L}^2(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$ . Then there exists a unique solution of problem (4.5)*

$$\begin{aligned} \mathbf{u} &\in L^2(0, T; \mathbf{H}^2(\Omega)) \cap C([0, T]; \mathbf{H}_0^1(\Omega)), \\ \mathbf{u}' &\in L^2(0, T; \mathbf{L}^2(\Omega)). \end{aligned}$$

Moreover, this solution satisfies the estimate

$$\begin{aligned} \frac{\alpha}{2} \|\mathbf{u}'\|_{L^2(0, T; \mathbf{L}^2(\Omega))}^2 + \frac{(1 - \omega)^2}{2} \|A \mathbf{u}\|_{L^2(0, T; \mathbf{L}^2(\Omega))}^2 + (1 - \omega) \|D \mathbf{u}\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))}^2 \\ + (1 - \omega) \|\operatorname{div} \mathbf{u}\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))}^2 \leq 4(1 - \omega) \|D \mathbf{u}_0\|^2 + \|\mathfrak{F}\|_{L^2(0, T; \mathbf{L}^2(\Omega))}^2. \end{aligned} \quad (4.6)$$

**Proof.**

By classical result of Agmon-Douglis-Nirenberg [1],  $A = -\Delta - \nabla \operatorname{div}$  is a strongly elliptic

operator, and generates an analytic semigroup in  $\mathbf{L}^2(\Omega)$  with domain  $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$  (we can see for instance [8]).

We start by showing the estimate (4.6). Multiply (4.5)<sub>1</sub> in  $\mathbf{L}^2(\Omega)$  by  $\mathbf{u}' + \alpha(1 - \omega)A\mathbf{u}$ , then

$$\begin{aligned} \int_{\Omega} |\mathbf{u}'|^2 + 2(1 - \omega) \int_{\Omega} \mathbf{u}' \cdot A\mathbf{u} + (1 - \omega)^2 \int_{\Omega} |A\mathbf{u}|^2 \\ = \int_{\Omega} \mathfrak{F} \cdot \mathbf{u}' + (1 - \omega) \int_{\Omega} \mathfrak{F} \cdot A\mathbf{u}. \end{aligned}$$

Integrate by parts the second term, we obtain

$$\begin{aligned} \|\mathbf{u}'\|^2 + (1 - \omega) \frac{d}{dt} \left( \|D\mathbf{u}\|^2 + \|\operatorname{div} \mathbf{u}\|^2 \right) + (1 - \omega)^2 \|A\mathbf{u}\|^2 \leq \|\mathfrak{F}\| \cdot \|\mathbf{u}'\| \\ + \frac{(1 - \omega)}{4} \|\mathfrak{F}\| \cdot \|A\mathbf{u}\|. \end{aligned}$$

On the other hand, using Young's inequality on the two terms right, we get

$$\begin{aligned} \|\mathfrak{F}\| \cdot \|\mathbf{u}'\| &\leq \frac{1}{2} \|\mathfrak{F}\|^2 + \frac{1}{2} \|\mathbf{u}'\|^2, \\ (1 - \omega) \|\mathfrak{F}\| \cdot \|A\mathbf{u}\| &\leq \frac{1}{2} \|\mathfrak{F}\|^2 + \frac{(1 - \omega)^2}{2} \|A\mathbf{u}\|^2. \end{aligned}$$

Integrate over  $[0, T]$  and use the inequality

$$\|\operatorname{div} \mathbf{u}_0\| \leq 3 \|D\mathbf{u}_0\|,$$

then we get (4.6). ■

The second Lemma give some stronger estimates.

**Lemma 4.2** ([8, 6]). *Under the conditions of Lemma 4.1 and if  $\partial\Omega \in C^3$ ,  $\mathfrak{F} \in L^2(0, T; \mathbf{H}^1(\Omega))$ ,  $\mathfrak{F}' \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$  and  $\mathbf{u}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_0^1(\Omega)$ . Then the solution  $\mathbf{u}$  of problem (4.5) given by Lemma 4.1 is such that*

$$\begin{aligned} \mathbf{u} &\in L^2(0, T; \mathbf{H}^3(\Omega)) \cap C([0, T]; \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)), \\ \mathbf{u}' &\in L^2(0, T; \mathbf{H}_0^1(\Omega)) \cap C([0, T]; \mathbf{L}^2(\Omega)). \end{aligned}$$

and there exists a constant  $C_1$ , depend only in  $\Omega$ ,  $T$ ,  $\alpha$  and  $\omega$ , such that one has the estimate

$$\begin{aligned} \|\mathbf{u}\|_{L^2(0, T, \mathbf{H}^3(\Omega))}^2 + \|\mathbf{u}\|_{L^\infty(0, T, \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))}^2 + \|\mathbf{u}'\|_{L^2(0, T, \mathbf{H}_0^1(\Omega))}^2 + \|\mathbf{u}'\|_{L^\infty(0, T, \mathbf{L}^2(\Omega))}^2 \\ \leq C_1 \{ \|A\mathbf{u}_0\|^2 + \|\mathfrak{F}(0)\|^2 + \|\mathfrak{F}\|_{L^2(0, T, \mathbf{H}^1(\Omega))}^2 + \|\mathfrak{F}'\|_{L^2(0, T, \mathbf{H}^{-1}(\Omega))}^2 \}. \end{aligned} \quad (4.7)$$

**Proof.**

Derive in terms of  $t$  the equation (4.5)<sub>1</sub>, then we obtain

$$\mathbf{u}'' + \alpha(1 - \omega)A\mathbf{u}' = \mathfrak{F}', \quad \text{in } Q_T,$$



and  $\mathbf{u}'_{|\partial\Omega}(t) = 0$  for all  $t \in [0, T]$ . Let  $\mathbf{v} = \mathbf{u}'$ , then  $\mathbf{v}$  verify the system

$$\begin{cases} \mathbf{v}' + \alpha(1 - \omega)A\mathbf{v} &= \mathfrak{F}', & \text{in } Q_T, \\ \mathbf{v}(0) &= \mathbf{v}_0 = \mathfrak{F}(0) - \alpha(1 - \omega)A\mathbf{u}_0, & \text{in } \Omega, \\ \mathbf{v} &= 0, & \text{on } \Sigma_T. \end{cases} \quad (4.8)$$

Multiply by  $\mathbf{v}$  the equation (4.10)<sub>1</sub> and integrate on  $\Omega$ . It comes

$$\int_{\Omega} \mathbf{v}' \cdot \mathbf{v} + \alpha(1 - \omega) \int_{\Omega} A\mathbf{v} \cdot \mathbf{v} = \langle \mathfrak{F}', \mathbf{v} \rangle_{\mathbf{H}^{-1}, \mathbf{H}_0^1}.$$

After calculation, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}\|^2 + \frac{3\alpha(1 - \omega)}{4} \|D\mathbf{v}\|^2 + \alpha(1 - \omega) \|\operatorname{div} \mathbf{v}\|^2 \leq \frac{1}{\alpha(1 - \omega)} \|\mathfrak{F}'\|_{\mathbf{H}^{-1}}^2.$$

Integrate on  $[0, T]$  and replace  $\mathbf{v}$  and  $\mathbf{v}_0$  by their values

$$\begin{aligned} \frac{1}{2} \|\mathbf{u}'\|_{L^\infty(0, T, L^2(\Omega))}^2 + \frac{3\alpha(1 - \omega)}{4} \|D\mathbf{u}'\|_{L^2(0, T, L^2(\Omega))}^2 + \alpha(1 - \omega) \|\operatorname{div} \mathbf{u}'\|_{L^2(0, T, L^2(\Omega))}^2 \\ \leq \frac{1}{\alpha(1 - \omega)} \|\mathfrak{F}'\|_{L^2(0, T, \mathbf{H}^{-1}(\Omega))}^2 + \frac{1}{2} \left( \|\mathfrak{F}(0)\|^2 + \alpha(1 - \omega) \|A\mathbf{u}_0\|^2 \right). \end{aligned} \quad (4.9)$$

Finally, inequality (4.7) follows from inequality (4.6) and (4.9). ■

## 4.2 Resolution of the Transport Problems

We consider the following two linear transport problems,

$$\begin{cases} \sigma' + (\mathbf{w} \cdot \nabla)\sigma + \sigma \operatorname{div} \mathbf{w} &= -\varepsilon^{-2} \alpha \operatorname{div} \mathbf{w}, & \text{in } Q_T, \\ \sigma(0, \cdot) &= \sigma_0, & \text{in } \Omega, \end{cases} \quad (4.10)$$

and

$$\begin{cases} \tau + \operatorname{We} \left( \tau' + (\mathbf{w} \cdot \nabla)\tau + \mathbf{g}(\nabla \mathbf{w}, \tau) \right) &= 2\omega \mathbf{D}[\mathbf{w}], & \text{in } Q_T, \\ \tau(0, \cdot) &= \tau_0, & \text{in } \Omega, \end{cases} \quad (4.11)$$

where  $\sigma_0$  and  $\tau_0$  are, respectively, some given function and symmetric tensor defined in  $\Omega$ . The existence of solutions to this problems follows from the classical method of characteristics. (see for example [3, 6, 8]). The lemmas below give some estimates of the solutions of these problems.

**Lemma 4.3** ([8]). *Let  $\Gamma \in C^1$ ,  $\mathbf{w} \in L^1(0, T, \mathbf{H}^3(\Omega))$ ,  $\mathbf{w} \cdot \mathbf{n} = 0$  on  $\Sigma_T$ , and  $\sigma_0 \in \mathbf{H}^2(\Omega)$ , with  $\int_{\Omega} \sigma_0 d\mathbf{x} = 0$ . Then there exists a unique solution  $\sigma \in C([0, T]; \mathbf{H}^2(\Omega))$  of (4.10) such that*

$$\int_{\Omega} \sigma(\cdot, \mathbf{x}) d\mathbf{x} = 0 \quad \text{in } [0, T],$$

and satisfying the following estimate

$$\|\sigma\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} \leq (\|\sigma_0\| + \alpha\varepsilon^{-2}) \exp(C_\Omega \|\mathbf{w}\|_{L^1(0, T, \mathbf{H}^3(\Omega))}),$$

for some positive constant  $C_\Omega$  depending on  $\Omega$ .

If, in addition,  $\mathbf{w} \in C([0, T]; \mathbf{H}^2(\Omega))$ , then  $\sigma' \in C([0, T]; \mathbf{H}^1(\Omega))$  satisfies

$$\|\sigma'\|_{L^\infty(0, T, \mathbf{H}^1(\Omega))} \leq C_\Omega \|\mathbf{w}\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} (\|\sigma_0\| + \alpha \varepsilon^{-2}) \exp \left( C_\Omega \|\mathbf{w}\|_{L^1(0, T, \mathbf{H}^3(\Omega))} \right).$$

**Lemma 4.4** ([3]). Let  $\Omega \subset \mathbb{R}^3$  be a domain of class  $C^3$ ,  $\mathbf{w} \in L^1(0, T; \mathbf{H}^3(\Omega) \cap \mathbf{H}_0^1(\Omega))$  and  $\tau_0 \in \mathbf{H}^2(\Omega)$ . Then there exists a unique solution  $\tau \in C([0, T]; \mathbf{H}^2(\Omega))$  of (4.11), such that

$$\|\tau\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} \leq \left( \|\tau_0\|^2 + \frac{2\omega}{C_\Omega \text{We}} \right) \exp \left( C_\Omega \|\mathbf{w}\|_{L^1(0, T, \mathbf{H}^3(\Omega))} \right),$$

for some positive constant  $C_\Omega$  depending on  $\Omega$ .

If, in addition,  $\mathbf{w} \in C([0, T]; \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))$ , then  $\tau' \in C([0, T]; \mathbf{H}^1(\Omega))$  satisfies

$$\|\tau'\|_{L^\infty(0, T, \mathbf{H}^1(\Omega))} \leq C_0 \left( \|\mathbf{w}\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} + \frac{1}{C_\Omega \text{We}} \right) \left( \|\tau_0\| + \frac{2\omega}{C_\Omega \text{We}} \right) \exp \left( C_\Omega \|\mathbf{w}\|_{L^1(0, T, \mathbf{H}^3(\Omega))} \right).$$

### 4.3 Proof of Theorem 3.1

We are now in a position to prove the local existence of a solution to problem (3.1). We apply the Theorem of fixed-point of Schauder.

Take  $T > 0$ ,  $\mathfrak{B}_1, \mathfrak{B}_2 > 0$ , and define

$$\begin{aligned} \mathfrak{R}_T = \{ & (\mathbf{w}, \pi, \psi), \\ & \mathbf{w} \in C([0, T]; \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \cap L^\infty(0, T, \mathbf{H}^3(\Omega)), \mathbf{w}' \in C([0, T]; \mathbf{L}^2(\Omega)) \cap L^2(0, T, \mathbf{H}_0^1(\Omega)) \\ & \pi \in L^\infty(0, T, \mathbf{H}^2(\Omega)), \pi' \in L^\infty(0, T, \mathbf{H}^1(\Omega)), \\ & \psi \in L^\infty(0, T, \mathbf{H}^2(\Omega)), \psi' \in L^\infty(0, T, \mathbf{H}^1(\Omega)), \\ & \mathbf{w}(0) = \mathbf{u}_0, \pi(0) = \sigma_0, \psi(0) = \tau_0 \text{ in } \Omega, \mathbf{w} = 0 \text{ in } \Sigma_T, \\ & \|\mathbf{w}\|_{L^\infty(0, T, \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))}^2 + \|\mathbf{w}\|_{L^2(0, T, \mathbf{H}^3(\Omega))}^2 + \|\mathbf{w}'\|_{L^\infty(0, T, \mathbf{L}^2(\Omega))}^2 + \|\mathbf{w}'\|_{L^2(0, T, \mathbf{H}_0^1(\Omega))}^2 \leq \mathfrak{B}_1, \\ & \|\pi\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} + \|\psi\|_{L^\infty(0, T, \mathbf{H}^2(\Omega))} \leq \mathfrak{B}_1, \\ & \|\pi'\|_{L^\infty(0, T, \mathbf{H}^1(\Omega))} + \|\psi'\|_{L^\infty(0, T, \mathbf{H}^1(\Omega))} \leq \mathfrak{B}_2, \\ & \frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2 \pi(t, x) \leq 2\mathfrak{M}_1, \quad \text{in } \overline{Q}_T \}. \end{aligned}$$

Choose  $\mathfrak{B}_1$  such that

$$\mathfrak{B}_1 > \max\{C_4 \|A\mathbf{u}_0\|^2, \|\sigma_0\|_2, \|\tau_0\|_2\}, \quad (4.12)$$

then  $(\mathbf{u}_0, \sigma_0, \tau_0) \in \mathfrak{R}_T$ . In fact,  $\mathbf{w}$  is a solution of problem

$$\begin{cases} \mathbf{w}(\cdot) \in \mathbf{H}^1(\Omega), \\ \mathbf{w}' + (1 - \omega)A\mathbf{w} = 0, & \text{p.p. in } \mathbb{R}_+, \\ \mathbf{w}(0) = \mathbf{u}_0, & \text{in } \Omega, \\ \mathbf{w} = 0, & \text{on } \Sigma_T. \end{cases} \quad (4.13)$$

Using estimate (4.7), there exists a constant  $C_4$  such that

$$\|\mathbf{w}'\|_{L^2(0, T, \mathbf{H}^3(\Omega))}^2 + \|\mathbf{w}'\|_{L^\infty(0, T, \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))}^2 + \|\mathbf{w}\|_{L^2(0, T, \mathbf{H}_0^1(\Omega))}^2 + \|\mathbf{w}\|_{L^\infty(0, T, \mathbf{L}^2(\Omega))}^2 \leq C_4 \|A\mathbf{u}_0\|^2.$$

Thus the choose of  $\mathfrak{B}_1$  in (4.12) is enough for prove that  $\mathfrak{R}_T$  is non empty for each  $T > 0$ .

Define now the application mapping  $\mathfrak{K}$  in this way

$$\begin{aligned} \mathfrak{K} : \quad \mathfrak{R}_T &\longrightarrow \mathfrak{X}_T = C([0, T]; \mathbf{H}_0^1(\Omega)) \times C([0, T]; H^1(\Omega)) \times C([0, T]; \mathbf{H}^1(\Omega)) \\ (\mathbf{w}, \pi, \psi) &\longrightarrow (\mathbf{u}, \sigma, \tau) \end{aligned}$$

where  $\mathbf{u}$ ,  $\sigma$  and  $\tau$  are solution of (4.5), (4.10) and (4.11), respectively, with

$$\begin{aligned} \mathfrak{F} &= \alpha \mathbf{f} + (1 - \omega) \frac{\varepsilon^2 \pi}{\alpha + \varepsilon^2 \pi} A \mathbf{w} + \frac{\varepsilon^2}{\alpha + \varepsilon^2 \pi} (\pi - w(\pi)) \nabla \pi - \alpha (\mathbf{w} \cdot \nabla) \mathbf{w} - \nabla \pi + \mathbf{div} \psi, \\ \mathcal{G} &= -\varepsilon^{-2} \alpha \operatorname{div} \mathbf{w}. \end{aligned}$$

If we take

$$\begin{aligned} \mathfrak{B}_1 &> \max \left\{ C_4 \|A \mathbf{u}_0\|^2, e^{\sqrt{2}} \left( \|\sigma_0\|_2 + \|\tau_0\|_2 + 1 + \frac{2\omega}{C_3 \operatorname{We}} \right), C_2 (2C_5 + 1) \|A \mathbf{u}_0\|^2 \right. \\ &\quad + C_5 \|A \mathbf{u}_0\|^4 + 3 \left( 2(1 + \|w\|_{\mathcal{C}}^2) \|\sigma_0\|_1^2 + \|\tau_0\|_1^2 \right) \\ &\quad \left. + 3 \|\mathbf{f}(0)\|^2 + 3 \|\mathbf{f}\|_{L^2(0, T, H^1(\Omega))}^2 + 3 \|\mathbf{f}'\|_{L^2(0, T, H^1(\Omega))}^2 \right\}, \end{aligned} \quad (4.14)$$

and

$$\mathfrak{B}_2 > e^{\sqrt{2}} \left\{ C_6 \left( \|\sigma_0\|_2 + \|\tau_0\|_2 + 1 + \frac{2\omega}{C_3 \operatorname{We}} \right) + \frac{1}{\operatorname{We}} \left( \|\tau_0\|_2 + \frac{2\omega}{C_3 \operatorname{We}} \right) \right\}, \quad (4.15)$$

and for all  $T$  small enough such that

$$T \leq T^* = \min \left( \frac{\mathfrak{B}_1}{2C_2(4C_4(1 + \|w\|_{\mathcal{C}}^2)\mathfrak{B}_1^2 + 3\mathfrak{B}_2^2)}, \frac{2}{C_6^2 \mathfrak{B}_1} \right), \quad (4.16)$$

we have  $\mathfrak{K}(\mathfrak{R}_T) \subset \mathfrak{R}_T$ .

We now use Schauder fixed point theorem. The mapping  $\mathfrak{K}$  is defined from convex, bounded and no empty set  $\mathfrak{R}_T$  into  $\mathfrak{X}_T$ . To finish, we need to show the continuity of  $\mathfrak{K}$  in  $\mathfrak{X}_T$ .

**Lemma 4.5.** *To show the continuity of  $\mathfrak{K}$  in  $\mathfrak{X}_T$ , it is enough to show the continuity of  $\mathfrak{K}$  in*

$$\mathfrak{Y}_T = C([0, T]; \mathbf{L}^2(\Omega)) \times C([0, T]; L^2(\Omega)) \times C([0, T]; \mathbf{L}^2(\Omega)).$$

*Proof.* Let  $\left( (\mathbf{w}_n, \pi_n, \psi_n) \right)_n$  be a sequence of  $\mathfrak{R}_T$  and tends to  $(\mathbf{w}, \pi, \psi)$ , such that:

$$(\mathbf{u}_n, \sigma_n, \tau_n) = \mathfrak{K}(\mathbf{w}_n, \pi_n, \psi_n) \text{ and } (\mathbf{u}, \sigma, \tau) = \mathfrak{K}(\mathbf{w}, \pi, \psi).$$

Suppose that  $\mathfrak{K}$  is continuous in  $\mathfrak{Y}_T$ , then the sequence  $\left( \mathfrak{K}(\mathbf{w}_n, \pi_n, \psi_n) \right)_n$  tends to  $\mathfrak{K}(\mathbf{w}, \pi, \psi)$  in  $\mathfrak{Y}_T$ , i.e.

$$\lim_{n \rightarrow \infty} \|(\mathbf{u}_n, \sigma_n, \tau_n) - (\mathbf{u}, \sigma, \tau)\|_{\mathfrak{Y}_T} = 0. \quad (4.17)$$

$\mathfrak{R}_T$  is a compact set in  $\mathfrak{X}_T$  (see for instance [5]). Using (4.17), we can extract of  $\left( (\mathbf{u}_n, \sigma_n, \tau_n) \right)_n$  a subsequence converges in  $\mathfrak{X}_T$  to the unique accumulation point  $(\mathbf{u}, \sigma, \tau)$ . Then the sequence  $\left( (\mathbf{u}_n, \sigma_n, \tau_n) \right)_n = \left( \mathfrak{K}(\mathbf{w}_n, \pi_n, \psi_n) \right)_n$  converges to  $(\mathbf{u}, \sigma, \tau) = \mathfrak{K}(\mathbf{w}, \pi, \psi)$  in  $\mathfrak{X}_T$ . This proved the continuity of  $\mathfrak{K}$  in  $\mathfrak{X}_T$ .  $\square$

**Lemma 4.6.**  $\mathfrak{K}$  is continuous in  $\mathfrak{Y}_T$ .

**Proof.**

Let  $\left((\mathbf{w}_n, \pi_n, \psi_n)\right)_n$  be a sequence of  $\mathfrak{R}_T$  and tends to  $(\mathbf{w}, \pi, \psi)$ , such that:

$$(\mathbf{u}_n, \sigma_n, \tau_n) = \mathfrak{K}(\mathbf{w}_n, \pi_n, \psi_n) \text{ and } (\mathbf{u}, \sigma, \tau) = \mathfrak{K}(\mathbf{w}, \pi, \psi).$$

Consider two systems. The first is :

$$\left\{ \begin{array}{lcl} \alpha \mathbf{u}'_n + (1 - \omega) A \mathbf{u}_n & = & \mathfrak{F}_n, \\ \sigma'_n + (\mathbf{w}_n \cdot \nabla) \sigma_n + \sigma_n \operatorname{div} \mathbf{w}_n & = & \mathcal{G}_n, \\ \tau_n + \operatorname{We} \{ \tau'_n + (\mathbf{w}_n \cdot \nabla) \tau_n + \mathbf{g}(\nabla \mathbf{w}_n, \tau_n) \} & = & 2\omega \mathbf{D}[\mathbf{w}_n], \text{ in } Q_T, \\ \mathbf{u}_n(0, x) & = & \mathbf{u}_0(x), \\ \sigma_n(0, x) & = & \sigma_0(x), \\ \tau_n(0, x) & = & \tau_0(x), \quad \text{in } \Omega, \\ \mathbf{u}_n & = & 0, \quad \text{on } \Sigma_T, \end{array} \right. \quad (4.18)$$

with

$$\begin{aligned} \mathfrak{F}_n &= \mathbf{F}(\mathbf{w}_n, \pi_n, \psi_n) - \alpha(\mathbf{w}_n \cdot \nabla) \mathbf{w}_n - \nabla \pi_n + \operatorname{div} \psi_n, \\ \mathcal{G}_n &= -\varepsilon^{-2} \operatorname{div} \mathbf{w}_n. \end{aligned}$$

And, the second is:

$$\left\{ \begin{array}{lcl} \alpha \mathbf{u}' + (1 - \omega) A \mathbf{u} & = & \mathfrak{F}, \\ \sigma' + (\mathbf{w} \cdot \nabla) \sigma + \sigma \operatorname{div} \mathbf{w} & = & \mathcal{G}, \\ \tau + \operatorname{We} \{ \tau' + (\mathbf{w} \cdot \nabla) \tau + \mathbf{g}(\nabla \mathbf{w}, \tau) \} & = & 2\omega \mathbf{D}[\mathbf{w}], \text{ in } Q_T, \\ \mathbf{u}(0, x) & = & \mathbf{u}_0(x), \\ \sigma(0, x) & = & \sigma_0(x), \\ \tau(0, x) & = & \tau_0(x), \quad \text{in } \Omega, \\ \mathbf{u} & = & 0, \quad \text{on } \Sigma_T, \end{array} \right. \quad (4.19)$$

with

$$\begin{aligned} \mathfrak{F} &= \mathbf{F}(\mathbf{w}, \pi, \psi) - \alpha(\mathbf{w} \cdot \nabla) \mathbf{w} - \nabla \pi + \operatorname{div} \psi, \\ \mathcal{G} &= -\varepsilon^{-2} \operatorname{div} \mathbf{w}. \end{aligned}$$

Let  $\mathbf{v}_n = \mathbf{u}_n - \mathbf{u}$ ,  $q_n = \sigma_n - \sigma$  and  $\mathbf{S}_n = \tau_n - \tau$ . Using (4.18) and (4.19), we obtain, in  $Q_T$ :

$$\left\{ \begin{array}{lcl} \alpha \mathbf{v}'_n + (1 - \omega) A \mathbf{v}_n & = & \mathfrak{F}_1, \\ q'_n + (\mathbf{w}_n \cdot \nabla) q_n + q_n \operatorname{div} \mathbf{w}_n & = & \mathcal{G}_1 - ((\mathbf{w}_n - \mathbf{w}) \cdot \nabla) \sigma - \sigma \operatorname{div} (\mathbf{w}_n - \mathbf{w}), \\ \mathbf{S}_n + \operatorname{We} \{ \mathbf{S}'_n + (\mathbf{w}_n \cdot \nabla) \mathbf{S}_n + \mathbf{g}(\nabla \mathbf{w}_n, \mathbf{S}_n) \} & = & \mathcal{H}_1 - \operatorname{We} \{ ((\mathbf{w}_n - \mathbf{w}) \cdot \nabla) \tau + \mathbf{g}(\nabla (\mathbf{w}_n - \mathbf{w}), \tau) \}, \end{array} \right. \quad (4.20)$$

with the boundary conditions:

$$\left\{ \begin{array}{lcl} \mathbf{v}_n(0, x) & = & 0, \\ q_n(0, x) & = & 0, \\ \mathbf{S}_n(0, x) & = & 0, \text{ in } \Omega, \\ \mathbf{v}_n & = & 0, \text{ on } \Sigma_T, \end{array} \right. \quad (4.21)$$

such that:

$$\begin{aligned}\mathfrak{F}_1 &= \mathbf{F}(\mathbf{w}_n, \pi_n, \psi_n) - \mathbf{F}(\mathbf{w}, \pi, \psi) - \alpha \left[ (\mathbf{w}_n \cdot \nabla) \mathbf{w}_n - (\mathbf{w} \cdot \nabla) \mathbf{w} \right] - \nabla(\pi_n - \pi) + \mathbf{div}(\psi_n - \psi), \\ \mathcal{G}_1 &= -\varepsilon^{-2} \mathbf{div}(\mathbf{w}_n - \mathbf{w}), \\ \mathcal{H}_1 &= 2\omega \mathbf{D}[\mathbf{w}_n - \mathbf{w}].\end{aligned}$$

First, multiply the equation (4.20)<sub>1</sub> by  $\mathbf{v}_n$ , and integrate over  $\Omega$ . We get:

$$\begin{aligned}\alpha \frac{d}{dt} \|\mathbf{v}_n\|^2 + (1 - \omega) \left( \|\nabla \mathbf{v}_n\|^2 + \|\mathbf{div} \mathbf{v}_n\|^2 \right) &\leq \|\pi_n - \pi\|_1^2 + \|\psi_n - \psi\|_1^2 + 4 \|\mathbf{v}_n\|^2 \\ &\quad + \alpha^2 \|(\mathbf{w}_n \cdot \nabla) \mathbf{w}_n - (\mathbf{w} \cdot \nabla) \mathbf{w}\|^2 \\ &\quad + \|\mathbf{F}(\mathbf{w}_n, \pi_n, \psi_n) - \mathbf{F}(\mathbf{w}, \pi, \psi)\|^2\end{aligned}\quad (4.22)$$

We now estimate the term  $\|\mathbf{F}(\mathbf{w}_n, \pi_n, \psi_n) - \mathbf{F}(\mathbf{w}, \pi, \psi)\|^2$  on the right hand of (4.22). Using the two inequalities :

$$\frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2 \pi(t, x) \leq 2\mathfrak{M}_1 \text{ and } \frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2 \pi_n(t, x) \leq 2\mathfrak{M}_1,$$

we obtain:

$$\begin{aligned}\|\mathbf{F}(\mathbf{w}_n, \pi_n, \psi_n) - \mathbf{F}(\mathbf{w}, \pi, \psi)\|^2 &\leq C_7 \mathfrak{m}_1 \varepsilon^4 \left[ (1 - \omega)^2 \|\pi_n A \mathbf{w}_n - \pi A \mathbf{w}\|^2 + \|\pi_n \nabla \pi_n - \pi \nabla \pi\|^2 \right. \\ &\quad \left. + \|w(\pi_n) \nabla \pi_n - w(\pi) \nabla \pi\|^2 + \|\pi_n \mathbf{div} \mathbf{S}_n - \pi \mathbf{div} \mathbf{S}\|^2 \right].\end{aligned}$$

Then, (4.22) satisfies:

$$\alpha \frac{d}{dt} \|\mathbf{v}_n\|^2 + (1 - \omega) \left( \|\nabla \mathbf{v}_n\|^2 + \|\mathbf{div} \mathbf{v}_n\|^2 \right) \leq C_8 \ell_n + 4 \|\mathbf{v}_n\|^2, \quad (4.23)$$

with

$$\begin{aligned}\ell_n &= \|\pi_n - \pi\|_1^2 + \|\psi_n - \psi\|_1^2 + \alpha \|(\mathbf{w}_n \cdot \nabla) \mathbf{w}_n - (\mathbf{w} \cdot \nabla) \mathbf{w}\|^2 + (1 - \omega)^2 \|\pi_n A \mathbf{w}_n - \pi A \mathbf{w}\|^2 \\ &\quad + \|\pi_n \nabla \pi_n - \pi \nabla \pi\|^2 + \|w(\pi_n) \nabla \pi_n - w(\pi) \nabla \pi\|^2 + \|\pi_n \mathbf{div} \mathbf{S}_n - \pi \mathbf{div} \mathbf{S}\|^2.\end{aligned}$$

Second, multiply the equation (4.20)<sub>2</sub> by  $\varepsilon^2 q_n$  and integrate over  $\Omega$ . This yields:

$$\begin{aligned}\varepsilon^2 \frac{d}{dt} \|q_n\|^2 &\leq (1 + C_9 \|\sigma\|_2) \|\mathbf{w}_n - \mathbf{w}\|_1^2 + (1 + C_{10} \|\mathbf{w}_n\|_3) \|q_n\|^2 \\ &\leq (1 + C_9 \|\sigma\|_2) \|\mathbf{w}_n - \mathbf{w}\|_1^2 + j_n \|q_n\|^2,\end{aligned}\quad (4.24)$$

with  $j_n = 1 + C_{10} \|\mathbf{w}_n\|_3$ .

Finally, multiply the equation (4.20)<sub>3</sub> by  $\mathbf{S}_n/2\omega$ , we obtain

$$\begin{aligned}\frac{\text{We}}{2\omega} \frac{d}{dt} \|\mathbf{S}_n\|^2 &\leq \left( 1 + \frac{\text{We} C_{11}}{2\omega} \|\tau\|_2 \right) \|\mathbf{w}_n - \mathbf{w}\|_1^2 + \left( 1 + \frac{1}{2\omega} + \frac{\text{We} C_{12}}{2\omega} \|\mathbf{w}_n\|_3 \right) \|\mathbf{S}_n\|^2 \\ &\leq \left( 1 + \frac{\text{We} C_{11}}{2\omega} \|\tau\|_2 \right) \|\mathbf{w}_n - \mathbf{w}\|_1^2 + k_n \|\mathbf{S}_n\|^2,\end{aligned}\quad (4.25)$$

with  $k_n = 1 + \frac{1}{2\omega} + \frac{\text{We} C_{12}}{2\omega} \|\mathbf{w}_n\|_3$ .

The functions  $j_n$  and  $k_n$ , are positive and, because of the class of solutions we consider,  $j_n$  and  $k_n$  belong to  $L^1(0, T)$ . Therefore, by using (4.23), (4.24), and (4.25), we deduce from Gronwall's lemma that:

$$\|\mathbf{v}_n\|^2 \leq \frac{C_8}{\alpha} \int_0^t \exp\left(\frac{-4s}{\alpha}\right) \ell_n(s) ds, \quad (4.26)$$

$$\|q_n\|^2 \leq \frac{1}{\varepsilon^2} (dir0o + C_9 \mathfrak{B}_1) \int_s^t \exp\left(\int_0^s j_n(r) dr\right) \|\mathbf{w}_n(s) - \mathbf{w}(s)\|_1^2 ds, \quad (4.27)$$

$$\|\mathbf{S}_n\|^2 \leq \left(\frac{2\omega}{We} + C_{11} \mathfrak{B}_1\right) \int_s^t \exp\left(\int_0^s k_n(r) dr\right) \|\mathbf{w}_n(s) - \mathbf{w}(s)\|_1^2 ds. \quad (4.28)$$

The sequence  $\left((\mathbf{w}_n, \pi_n, \psi_n)\right)_n$  of  $\mathfrak{R}_T$  tends to  $(\mathbf{w}, \pi, \psi)$ , and using (4.26), (4.27) and (4.28), we obtain  $\mathbf{v}_n$ ,  $q_n$  and  $\mathbf{S}_n$  tend to zero in  $\mathfrak{Y}_T$ . This meaning that the sequence  $\left((\mathbf{u}_n, \sigma_n, \tau_n)\right)_n = \left(\mathfrak{K}(\mathbf{w}_n, \pi_n, \psi_n)\right)_n$  tends to  $(\mathbf{u}, \sigma, \tau) = \mathfrak{K}(\mathbf{w}, \pi, \psi)$  and  $\mathfrak{K}$  is continuous in  $\mathfrak{Y}_T$ . ■

#### 4.4 Proof of Theorem 3.2

We take, as usual, the difference of two solutions  $(\mathbf{u}_1, \sigma_1, \tau_1)$  and  $(\mathbf{u}_2, \sigma_2, \tau_2)$  belonging to the class specified in the theorem 3.2. The vector function  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ , the scalar function  $\sigma = \sigma_1 - \sigma_2$  and the tensor function  $\tau = \tau_1 - \tau_2$  satisfy the following system:

$$\begin{cases} \alpha [\mathbf{u}' + (\mathbf{u} \cdot \nabla) \mathbf{u}_1 + (\mathbf{u}_2 \cdot \nabla) \mathbf{u}] + (1 - \omega) A \mathbf{u} &= \mathfrak{F}_2 - \nabla \sigma + \mathbf{div} \tau, \\ \sigma' + (\mathbf{u}_1 \cdot \nabla) \sigma + (\mathbf{u} \cdot \nabla) \sigma_2 + \sigma \mathbf{div} \mathbf{u}_1 + \sigma_2 \mathbf{div} \mathbf{u} &= -\varepsilon^{-2} \mathbf{div} \mathbf{u}, \\ \tau + We \{ \tau' + (\mathbf{u}_1 \cdot \nabla) \tau + (\mathbf{u} \cdot \nabla) \tau_2 + \mathbf{g}(\nabla \mathbf{u}_1, \tau) + \mathbf{g}(\nabla \mathbf{u}, \tau_2) \} &= 2\omega \mathbf{D}[\mathbf{u}], \end{cases} \quad (4.29)$$

with the boundary conditions:

$$\begin{cases} \mathbf{u}(0, x) &= 0, \\ \sigma(0, x) &= 0, \\ \tau(0, x) &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \Sigma_T, \end{cases} \quad (4.30)$$

such that:

$$\mathfrak{F}_2 = \mathbf{F}(\mathbf{u}_1, \sigma_1, \tau_1) - \mathbf{F}(\mathbf{u}_2, \sigma_2, \tau_2).$$

Multiply (4.29)<sub>1</sub>, (4.29)<sub>2</sub> and (4.29)<sub>3</sub> by  $\mathbf{u}$ ,  $\varepsilon^2 \sigma / \alpha$ , and  $\tau / (2\omega)$ , respectively, and integrate over  $\Omega$ . Summing the three obtained equations, one obtains

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \alpha \|\mathbf{u}\|^2 + \frac{\varepsilon^2}{\alpha} \|\sigma\|^2 + \frac{We}{2\omega} \|\tau\|^2 \right) + (1 - \omega) \left( \|\nabla \mathbf{u}\|^2 + \|\mathbf{div} \mathbf{u}\|^2 \right) + \frac{1}{2\omega} \|\tau\|^2 \\ & \leq \alpha C_{12} \left[ \|\mathbf{u}_1\| \|\mathbf{u}\|^2 + \|\mathbf{u}_2\| \|\mathbf{u}\|^2 \right] + \frac{\varepsilon^2}{\alpha} C_{12} \left[ \|\sigma\| \|\mathbf{u}_1\|_3 \|\nabla \mathbf{u}\| + \|\sigma_2\|_2 \|\nabla \mathbf{u}\| \|\mathbf{u}\| \right. \\ & \quad \left. + \|\sigma\| \|\sigma_1\|_2 \|\mathbf{div} \mathbf{u}\| + \|\sigma_2\|_2 \|\sigma\| \|\mathbf{div} \mathbf{u}\| + \|\tau_1\| \|\sigma\| \|\nabla \mathbf{u}\| + \|\sigma_2\| \|\tau\| \|\nabla \mathbf{u}\| \right] \\ & \quad + \frac{\varepsilon^2}{\alpha} C_{12} \left[ \|\mathbf{u}_1\|_3 \|\sigma\|^2 + \|\nabla \mathbf{u}\| \|\sigma_2\|_2 \|\sigma\| \right] + \frac{We}{2\omega} C_{12} \left[ \|\mathbf{u}_1\|_3 \|\tau\|^2 + \|\nabla \mathbf{u}\| \|\tau_2\|_2 \|\tau\| \right]. \end{aligned} \quad (4.31)$$

For  $\delta > 0$ , (4.31) can be written as:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( \alpha \|\mathbf{u}\|^2 + \frac{\varepsilon^2}{\alpha} \|\sigma\|^2 + \frac{\text{We}}{2\omega} \|\tau\|^2 \right) + (1-\omega) \left( \|\nabla \mathbf{u}\|^2 + \|\text{div } \mathbf{u}\|^2 \right) + \frac{1}{2\omega} \|\tau\|^2 \\
& \leq \left[ C_{12} (\|\mathbf{u}_1\| + \|\mathbf{u}_2\|) + \frac{(C_{12})^2}{2\delta} \|\sigma_2\|_2^2 \right] \alpha \|\mathbf{u}\|^2 \\
& \quad + \frac{\delta}{2} \left[ \left( \frac{5\varepsilon^2}{\alpha} + \frac{\text{We}}{2\omega} \right) \|\nabla \mathbf{u}\|^2 + \frac{2\varepsilon^2}{\alpha} \|\text{div } \mathbf{u}\|^2 \right] \\
& \quad + \left[ \frac{(C_{12})^2}{2\delta} \left( \|\mathbf{u}_1\|_2^3 + \|\sigma_1\|_2^2 + 2\|\sigma_2\|_2^2 + \|\tau_1\|_2^2 \right) + C_{12} \|\mathbf{u}_1\|_3 \right] \frac{\varepsilon^2}{\alpha} \|\sigma\|^2 \\
& \quad + \left[ \frac{(C_{12})^2}{2\delta} \|\sigma_2\|_2^2 + C_{12} \|\mathbf{u}_1\|_3 \right] \frac{\text{We}}{2\omega} \|\tau\|^2. \tag{4.32}
\end{aligned}$$

From (4.32), we then deduce that solutions  $(\mathbf{u}, \sigma, \tau)$  of (4.29) satisfy the following energy inequality:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( \alpha \|\mathbf{u}\|^2 + \frac{\varepsilon^2}{\alpha} \|\sigma\|^2 + \frac{\text{We}}{2\omega} \|\tau\|^2 \right) + (1-\omega) \left( 1 - \frac{\delta(10\varepsilon^2\omega + \alpha\text{We})}{4\alpha\omega(1-\omega)} \right) \|\nabla \mathbf{u}\|^2 \\
& \quad + (1-\omega) \left( 1 - \frac{\delta\varepsilon^2}{\alpha(1-\omega)} \right) \|\text{div } \mathbf{u}\|^2 + \frac{1}{2\omega} \|\tau\|^2 \leq \mathcal{X}_\delta \left[ \alpha \|\mathbf{u}\|^2 + \frac{\varepsilon^2}{\alpha} \|\sigma\|^2 + \frac{\text{We}}{2\omega} \|\tau\|^2 \right], \tag{4.33}
\end{aligned}$$

with

$$\mathcal{X}_\delta = C_{12} (\|\mathbf{u}_1\| + \|\mathbf{u}_2\| + \|\mathbf{u}_1\|_3) + \frac{(C_{12})^2}{2\delta} \left( \|\mathbf{u}_1\|_2^3 + \|\sigma_1\|_2^2 + 2\|\sigma_2\|_2^2 + \|\tau_1\|_2^2 \right). \tag{4.34}$$

The function  $\mathcal{X}_\delta$ , defined in (4.34), is positive. Moreover, because of the class of solutions we consider,  $\mathcal{X}_\delta$  belongs to  $L^1(0, T)$ . Therefore, choosing  $\delta > 0$  small enough, we deduce from Gronwall's lemma that  $\mathbf{u} = 0$ ,  $\sigma = 0$  and  $\tau = 0$  in  $Q_T$ , and that consequently  $\mathbf{u}_1 = \mathbf{u}_2$ ,  $\sigma_1 = \sigma_2$ ,  $\tau_1 = \tau_2$  in  $Q_T$  and the system (4.29) has a unique solution.

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